

## "Hardy-Sobolev inequalities for vector fields and canceling linear differential operators"

Bousquet, Pierre ; Van Schaftingen, Jean

### Abstract

Given a homogeneous  $k$ -th order differential operator  $A(D)$  on  $\mathbb{R}^n$  between two finite dimensional spaces, we establish the Hardy inequality  $\int_{\mathbb{R}^n} \frac{|D^{k-1}u|^2}{|x|^2} dx \leq C \int_{\mathbb{R}^n} |A(D)u|^2 dx$  and the Sobolev inequality  $\|D^{k-n}u\|_{L^\infty(\mathbb{R}^n)} \leq C \int_{\mathbb{R}^n} |A(D)u|^2 dx$  when  $A(D)$  is elliptic and satisfies a recently introduced cancellation property. We also study the necessity of these two conditions.

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# HARDY–SOBOLEV INEQUALITIES FOR VECTOR FIELDS AND CANCELING LINEAR DIFFERENTIAL OPERATORS

PIERRE BOUSQUET AND JEAN VAN SCHAFTINGEN

ABSTRACT. Given a homogeneous  $k$ -th order differential operator  $A(D)$  on  $\mathbf{R}^n$  between two finite dimensional spaces, we establish the Hardy inequality

$$\int_{\mathbf{R}^n} \frac{|D^{k-1}u|^2}{|x|^2} dx \leq C \int_{\mathbf{R}^n} |A(D)u|^2$$

and the Sobolev inequality

$$\|D^{k-n}u\|_{L^\infty(\mathbf{R}^n)} \leq C \int_{\mathbf{R}^n} |A(D)u|^2$$

when  $A(D)$  is elliptic and satisfies a recently introduced cancellation property. We also study the necessity of these two conditions.

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## 1. INTRODUCTION

Let  $k \in \mathbf{N}_*$  and let  $V, E$  be two real finite dimensional vector spaces. Given a homogeneous  $k$ -th order differential operator  $A(D)$  on  $\mathbf{R}^n$  from  $V$  to  $E$ , we address the question of controlling any vector field  $u \in C^\infty(\mathbf{R}^n; V)$  by the vector field  $A(D)u$ , where the vector differential operator  $A(D)$  is defined by

$$A(D)u = \sum_{\substack{\alpha \in \mathbf{N}^n \\ |\alpha|=k}} A_\alpha [\partial^\alpha u] \in C^\infty(\mathbf{R}^n; E).$$

Here,  $A_\alpha$  is a linear map in  $\mathcal{L}(V; E)$ , for every  $\alpha \in \mathbf{N}^n$  with  $|\alpha| = k$ .

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The classical theory of A. P. Calderón and A. Zygmund asserts that for every  $p \in (1, \infty)$ , there exists  $C > 0$  such that for each compactly supported smooth vector field  $u \in C_c^\infty(\mathbf{R}^n; V)$ ,

$$\int_{\mathbf{R}^n} |D^k u|^p \leq C \int_{\mathbf{R}^n} |A(D)u|^p$$

if and only if the operator  $A(D)$  is *elliptic* [11], that is for every  $\xi \in \mathbf{R}^n \setminus \{0\}$ , the linear map  $A(\xi) := \sum_{\alpha \in \mathbf{N}^n, |\alpha|=k} \xi^\alpha A_\alpha \in \mathcal{L}(V; E)$  is one-to-one [1, §7; 2, definition 6.3; 13, theorem 1; 23, definition 1.7.1]. Examples of first-order homogeneous elliptic operators are given for  $V = \mathbf{R}$  by the gradient operator  $A(D)u = \nabla u$  and for  $V = \wedge^\ell \mathbf{R}^n$  by the exterior differential and codifferential  $A(D)u = (du, d^*u)$ .

The situation is dramatically different for  $p = 1$  as there is no nontrivial estimate of the  $L^1$ -norm of some component of  $D^k u$  by  $\int_{\mathbf{R}^n} |A(D)u|$  [15, 16, 22]. This does not end the story however. Even if the quantity  $\int_{\mathbf{R}^n} |A(D)u|$  is strictly weaker than  $\int_{\mathbf{R}^n} |Du|$ , it might still be possible to replace the latter by the former in some inequalities.

A first inequality to which this programme was applied is the *Gagliardo–Nirenberg–Sobolev inequality* [12, 21]

$$\left( \int_{\mathbf{R}^n} |u|^{\frac{n}{n-1}} \right)^{1-\frac{1}{n}} \leq C \int_{\mathbf{R}^n} |Du|. \quad (1.1)$$

The elliptic operators that can replace the derivative were characterized by a new cancellation condition.

**Theorem 1** (Van Schaftingen [30]). *Let  $A(D)$  be an elliptic homogeneous linear differential operator of order  $k$  on  $\mathbf{R}^n$  from  $V$  to  $E$  and  $\ell \in \{1, \dots, \min(k, n-1)\}$ . The estimate*

$$\left( \int_{\mathbf{R}^n} |D^{k-\ell} u|^{\frac{n}{n-\ell}} \right)^{1-\frac{\ell}{n}} \leq C \int_{\mathbf{R}^n} |A(D)u|,$$

*holds for every  $u \in C_c^\infty(\mathbf{R}^n; V)$  if and only if  $A(D)$  is canceling.*

The new cancellation condition was defined as

**Definition 1.1.** A homogeneous linear differential operator  $A(D)$  on  $\mathbf{R}^n$  from  $V$  to  $E$  is *canceling* if

$$\bigcap_{\xi \in \mathbf{R}^n \setminus \{0\}} A(\xi)[V] = \{0\}.$$

Theorem 1 covers in particular the classical inequality (1.1), the Hodge–Sobolev inequality of J. Bourgain and H. Brezis, and L. Lanzani and E. Stein [6, 7, 17] (see also [4, 5, 8, 27, 28]) and the Korn–Sobolev inequality [25].

In the present work we continue this programme for other classical inequalities in Sobolev spaces. We begin with the classical *Hardy inequality*: given  $n \geq 2$  and  $k \geq 1$ , there exists  $C > 0$  such that for every  $u \in C_c^\infty(\mathbf{R}^n; V)$ ,

$$\int_{\mathbf{R}^n} \frac{|D^{k-1}u(x)|}{|x|} dx \leq C \int_{\mathbf{R}^n} |D^k u|; \quad (1.2)$$

and we address the validity of the following inequality

$$\int_{\mathbf{R}^n} \frac{|D^{k-1}u(x)|}{|x|} dx \leq C \int_{\mathbf{R}^n} |A(D)u|. \quad (1.3)$$

Remarkably, it also depends on the cancellation condition.

**Theorem 2.** *Let  $A(D)$  be an elliptic homogeneous linear differential operator of order  $k$  on  $\mathbf{R}^n$  from  $V$  to  $E$  and  $\ell \in \{1, \dots, \min(k, n-1)\}$ . The estimate*

$$\int_{\mathbf{R}^n} \frac{|D^{k-\ell}u(x)|}{|x|^\ell} dx \leq C \int_{\mathbf{R}^n} |A(D)u|,$$

*holds for every  $u \in C_c^\infty(\mathbf{R}^n; V)$  if and only if  $A(D)$  is canceling.*

As particular cases of theorem 2, we have the classical Hardy inequality (1.2), the inequality of V. Maz'ya [9, 19]: for every  $u \in C_c^\infty(\mathbf{R}^n; \mathbf{R}^n)$ ,

$$\int_{\mathbf{R}^n} \frac{|Du(x)|}{|x|} dx \leq C \int_{\mathbf{R}^n} |\Delta u| + |\nabla \operatorname{div} u|. \quad (1.4)$$

a new Hodge–Hardy inequality: for every  $u \in C_c^\infty(\mathbf{R}^n; \wedge^\ell \mathbf{R}^n)$ , if  $2 \leq \ell \leq n-2$ ,

$$\int_{\mathbf{R}^n} \frac{|u(x)|}{|x|} dx \leq C \int_{\mathbf{R}^n} |du| + |d^*u|,$$

and a new Korn–Hardy inequality: for every  $u \in C_c^\infty(\mathbf{R}^n)$ ,

$$\int_{\mathbf{R}^n} \frac{|u(x)|}{|x|} dx \leq C \int_{\mathbf{R}^n} |\nabla^s u|,$$

where the symmetric derivative is defined as  $\nabla^s u(x) = \frac{1}{2}(Du(x) + (Du(x))^*)$ .

The proof of the sufficiency of the cancellation in theorem 2 is quite different from its counterpart in theorem 1. It combines in an original way the strategy of Bousquet and Mironescu [9] with algebraic properties of canceling operators [30] and properties of Green functions [14].

The second inequality that we study is the limiting Sobolev inequality (see for example [10, chapter 9, remark 13])

$$\sup_{x \in \mathbf{R}^n} |u(x)| \leq C \int_{\mathbf{R}^n} |D^n u|.$$

We prove a limiting case of theorem 1 that was left open [30, open problem 8.4]. Again, the cancellation property plays a role.

**Theorem 3.** *Let  $A(D)$  be a homogeneous linear differential operator of order  $k \geq n$  on  $\mathbf{R}^n$  from  $V$  to  $E$ . If  $A(D)$  is elliptic and canceling, then the estimate*

$$\sup_{x \in \mathbf{R}^n} |D^{k-n}u(x)| \leq C \int_{\mathbf{R}^n} |A(D)u|,$$

*holds for every  $u \in C_c^\infty(\mathbf{R}^n; V)$ .*

Theorem 3 covers in particular the estimate of P. Mironescu  $\|\nabla u\|_{L^\infty} \leq C \|\Delta^{\frac{n}{2}} \nabla u\|_{L^1}$  for  $n \in \mathbf{N}$  odd [20]. The proof of theorem 3 relies on theorem 2.

The cancellation is not necessary for the estimate of theorem 3 to hold [30, remark 5.1]: for example, the differentiation operator on  $\mathbf{R}$  is *not canceling*, but the inequality  $\|u\|_{L^\infty} \leq \|u'\|_{L^1}$  holds for every  $u \in C_c^\infty(\mathbf{R})$ .

In the scalar case  $\dim V = 1$ , the inequalities of theorems 1, 2 and 3 follow from the Sobolev embedding of  $W^{1,1}(\mathbf{R}^n)$  in the Lorentz space  $L^{\frac{n}{n-1},1}(\mathbf{R}^n)$  [3, 26]

$$\|u\|_{L^{\frac{n}{n-1},1}(\mathbf{R}^n)} \leq C \|Du\|_{L^1(\mathbf{R}^n)}.$$

It is not known whether this inequality can be extended to canceling operators [30, open problem 8.3] as

$$\|u\|_{L^{\frac{n}{n-1},1}(\mathbf{R}^n)} \leq C \|A(D)u\|_{L^1(\mathbf{R}^n)};$$

this inequality would be consistent with theorems 1, 2 and 3.

One can wonder whether the *ellipticity* is *necessary* in theorem 2 as it is in theorem 1  $\ell = 1$  [30, proposition 5.1]. In general, this is not the case. However, when  $\ell = 1$ , the ellipticity is necessary for a scale of Hardy-Sobolev inequalities.

**Theorem 4.** *Let  $A(D)$  be a homogeneous linear differential operator of order  $k$  on  $\mathbf{R}^n$  from  $V$  to  $E$  and let  $\lambda \in [0, 1)$ . The estimate*

$$\left( \int_{\mathbf{R}^n} \frac{|D^{k-1}u|^{\frac{n-\lambda}{n-1}}}{|x|^\lambda} dx \right)^{\frac{n-1}{n-\lambda}} \leq C \int_{\mathbf{R}^n} |A(D)u|,$$

*holds for every  $u \in C_c^\infty(\mathbf{R}^n; V)$  if and only if  $A(D)$  is elliptic and canceling.*

This result is already known for  $\lambda = 0$  [30]. For  $\lambda \in (0, 1)$ , the sufficiency part is a consequence of theorems 1 and 2 by the Hölder inequality. Alternatively, it can be proved in a more direct way by using the same arguments as its counterpart in theorem 2 bypassing the more delicate proof of theorem 1.

In the limiting case  $\lambda = 1$  in theorem 4, the ellipticity condition is not necessary in theorem 2. This phenomenon can already be observed in the scalar case: for every  $u \in C_c^\infty(\mathbf{R}^3)$ , one has

$$\int_{\mathbf{R}^3} \frac{|u(x)|}{|x|} dx \leq C \int_{\mathbf{R}^3} |\partial_1 u| + |\partial_2 u|, \quad (1.5)$$

and the operator  $(\partial_1, \partial_2)$  is not elliptic.

We give some partial results concerning the Hardy inequality (1.3) when the operator  $A(D)$  is not elliptic. First the cone

$$\{\xi \in \mathbf{R}^n : A(\xi) \text{ is not one-to-one}\}$$

should not be too large: for example, it cannot contain a hyperplane. In particular, the ellipticity condition turns out to be necessary when  $n = 2$ . The general problem of writing necessary and sufficient conditions on  $A(D)$  seems quite difficult, we have however written in theorem 5.1 such a condition for an operator  $A(D)$  which is a collection of components of first order derivatives:

$$A(D)u(x) = (a_1 \cdot Du(x)[b_1], \dots, a_\ell \cdot Du(x)[b_\ell]).$$

## 2. PROOF OF THE HARDY-SOBOLEV INEQUALITY

In this section, we prove the sufficiency part of theorem 2 and theorem 4. The following proposition gives in fact a more general result:

**Proposition 2.1.** *Let  $A(D)$  be a linear differential operator of order  $k$  on  $\mathbf{R}^n$  from  $V$  to  $E$  and  $\ell \in \{1, \dots, \min(k, n-1)\}$ . If  $A(D)$  is elliptic and canceling, then for every  $u \in C_c^\infty(\mathbf{R}^n; V)$  and  $q \in [1, \frac{n}{n-\ell})$ , there exists  $C \in \mathbf{R}$  such that*

$$\left( \int_{\mathbf{R}^n} \frac{|D^{k-\ell}u(x)|^q}{|x|^{n-(n-\ell)q}} dx \right)^{\frac{1}{q}} \leq C \int_{\mathbf{R}^n} |A(D)u|.$$

The first tool that we shall use is the existence of a Green function that allows to recover  $D^{k-\ell}$  from  $A(D)$ .

**Lemma 2.2.** *Let  $A(D)$  be a linear differential operator of order  $k$  on  $\mathbf{R}^n$  from  $V$  to  $E$ . If  $A(D)$  is elliptic and  $\ell \in \{1, \dots, \min(k, n-1)\}$ , then there exists a function  $G \in C^\infty(\mathbf{R}^n \setminus \{0\}; \mathcal{L}(E; \mathcal{L}_{k-\ell}(\mathbf{R}^n; V)))$  such that  $G$  is homogeneous of degree  $\ell - n$  and for every  $u \in C_c^\infty(\mathbf{R}^n; V)$  and  $x \in \mathbf{R}^n$ ,*

$$D^{k-\ell}u(x) = \int_{\mathbf{R}^n} G(x-y)[A(D)u(y)] \, dy.$$

Moreover,

$$\bigcap_{x \in \mathbf{R}^n \setminus \{0\}} \ker G(x) = \bigcap_{\xi \in \mathbf{R}^n \setminus \{0\}} \ker A(\xi)^*.$$

Here,  $\mathcal{L}_{k-\ell}(\mathbf{R}^n; V)$  is the space of  $(k-\ell)$ -linear maps from  $\mathbf{R}^n$  into  $V$ , and simply  $V$  when  $\ell = k$ . Here and in the sequel, we endow  $V$  and  $E$  with an inner product denoted by  $\cdot$  and the adjoint is taken with respect to that fixed Euclidean structure.

The restriction  $\ell \leq n-1$  is essential as it can be observed when  $n = 2$  and  $A(D) = \Delta$ . In that case, the Green function  $G$  is not homogeneous of degree 0.

We use the following convention to define the Fourier transform of a map  $u \in C_c^\infty(\mathbf{R}^n; V)$ :

$$\widehat{u}(\xi) = \int_{\mathbf{R}^n} u(x) e^{-i\xi \cdot x} \, dx;$$

the map  $\widehat{u}$  takes its values in the *complexified* vector space  $V \otimes \mathbf{C}$ .

*Proof.* We define the map  $H : \mathbf{R}^n \setminus \{0\} \rightarrow \mathcal{L}(E; \mathcal{L}_{k-\ell}(\mathbf{R}^n; V))$ , for every  $\xi \in \mathbf{R}^n \setminus \{0\}$  by

$$H(\xi)[e][v_1, \dots, v_{k-\ell}] = (\xi \cdot v_1) \cdots (\xi \cdot v_{k-\ell}) (A(\xi)^* \circ A(\xi))^{-1} [A(\xi)^*[e]].$$

The map  $H$  is smooth in  $\mathbf{R}^n \setminus \{0\}$  and is homogeneous of degree  $-\ell > -n$ . Hence,  $H$  defines a distribution on  $\mathbf{R}^n$  that we still denote by  $H$  and which is homogeneous of degree  $-\ell$  (see for example [14, theorem 3.2.3]). It follows that  $H$  is a temperate distribution. Moreover, the map  $G : \mathbf{R}^n \setminus \{0\} \rightarrow \mathcal{L}(E; \mathcal{L}_{k-\ell}(\mathbf{R}^n; V))$  defined by its Fourier transform  $\widehat{G} = i^{-\ell} H$ , is smooth in  $\mathbf{R}^n \setminus \{0\}$  [14, theorem 7.1.18] and homogeneous of degree  $\ell - n$  [14, theorem 7.1.16]. Finally, since for every  $\xi \in \mathbf{R}^n \setminus \{0\}$ ,  $(\widehat{A(D)u})(\xi) = i^k A(\xi) \widehat{u}(\xi)$ ,  $G$  satisfies the required identity by definition of  $H$ .  $\square$

The second ingredient is a duality estimate on  $A(D)u$ .

**Lemma 2.3.** *Let  $A(D)$  be a linear differential operator of order  $k$  on  $\mathbf{R}^n$  from  $V$  to  $E$ . If  $A(D)$  is elliptic and canceling, then there exists  $C \in \mathbf{R}$  and  $m \in \mathbf{N}_*$  such that for every  $u \in C_c^\infty(\mathbf{R}^n; V)$  and every  $\varphi \in C_c^\infty(\mathbf{R}^n; E)$ ,*

$$\left| \int_{\mathbf{R}^n} \varphi \cdot A(D)u \right| \leq C \sum_{j=1}^m \int_{\mathbf{R}^n} |A(D)u(x)| |x|^j |D^j \varphi(x)| \, dx.$$

The integer  $m$  that appears in the conclusion depends on  $A(D)$  and not only on its order; a rather pessimistic upper bound for  $m$  is  $2k \dim V$  [30, remark 4.1].

The proof of this lemma is similar to the proof of [30, proposition 8.9]. The idea of the integration by parts already appeared in the context of divergence-free vector fields [7, theorem 3; 9; 19, theorem 2; 29, lemma 4.5].

*Proof of lemma 2.3.* Since  $A(D)$  is canceling and elliptic there exist a finite dimensional vector space  $F$  and a homogeneous linear differential operator  $L(D)$  of order  $m \in \mathbf{N}_*$  from  $E$  to  $F$  such that  $L(D) \circ A(D) = 0$  and  $L(D)$  is cocanceling [30, proposition 4.2], that is,

$$\bigcap_{\xi \in \mathbf{R}^n \setminus \{0\}} \ker L(\xi) = \{0\}.$$

Writing  $L(D) = \sum_{|\alpha|=m} L_\alpha \partial^\alpha$ , by classical properties of linear operators, there exist  $K_\alpha \in \mathcal{L}(F; E)$  for  $\alpha \in \mathbf{N}^n$  with  $|\alpha| = m$  such that

$$\sum_{|\alpha|=m} K_\alpha \circ L_\alpha = \text{id}_E \quad (2.1)$$

(a detailed proof has been given in [30, lemma 2.5]). We define now, the homogeneous polynomial  $P : \mathbf{R}^n \rightarrow \mathcal{L}(E; F)$  of degree  $m$  for  $x \in \mathbf{R}^n$  by

$$P(x) = \sum_{|\alpha|=m} \frac{x^\alpha}{\alpha!} K_\alpha^*.$$

By the identity (2.1), we compute

$$L(D)^*(P) = \sum_{|\alpha|=m} L_\alpha^* \circ \partial^\alpha P = \sum_{|\alpha|=m} L_\alpha^* \circ K_\alpha^* = \text{id}_E^* = \text{id}_E.$$

For every  $u \in C_c^\infty(\mathbf{R}^n; V)$  and  $\varphi \in C_c^\infty(\mathbf{R}^n; E)$ , since  $L(D)[A(D)u] = 0$ , we get

$$\begin{aligned} \int_{\mathbf{R}^n} \varphi \cdot A(D)u &= \int_{\mathbf{R}^n} A(D)u \cdot (L(D)^*(P))[\varphi] \\ &= \int_{\mathbf{R}^n} A(D)u \cdot ((L(D)^*(P))[\varphi] - L(D)^*(P[\varphi])) . \end{aligned}$$

In order to conclude, we note that there exists  $C > 0$  such that for every  $x \in \mathbf{R}^n$ ,

$$|((L(D)^*P)[\varphi(x)] - L(D)^*(P[\varphi])(x))| \leq C \left( \sum_{j=1}^m |x|^j |D^j \varphi(x)| \right). \quad \square$$

We can now prove the main result of this section.

*Proof of proposition 2.1.* Let  $G$  be the Green function given by lemma 2.2. We choose  $\rho \in C_c^\infty(\mathbf{R}^n)$  such that  $\rho = 1$  on  $B_{1/4}$  and  $\text{supp } \rho \subset B_{1/2}$ , and we define  $G_1$  and  $G_2$  for  $x, y \in (\mathbf{R}^n \setminus \{0\}) \times \mathbf{R}^n$  with  $x \neq y$  by

$$G_1(x, y) = \rho\left(\frac{y}{|x|}\right) G(x)$$

and

$$G_2(x, y) = G(x - y) - \rho\left(\frac{y}{|x|}\right) G(x).$$

By lemma 2.3 and the homogeneity of  $G$ , we have

$$\begin{aligned} & \left( \int_{\mathbf{R}^n} \left| \int_{\mathbf{R}^n} G_1(x, y) [A(D)u(y)] dy \right|^q \frac{1}{|x|^{n-(n-\ell)q}} dx \right)^{\frac{1}{q}} \\ &= \left( \int_{\mathbf{R}^n} \left| \int_{\mathbf{R}^n} \rho\left(\frac{y}{|x|}\right) A(D)u(y) dy \right|^q \frac{|G(x)|^q}{|x|^{n-(n-\ell)q}} dx \right)^{\frac{1}{q}} \\ &\leq C \left( \int_{\mathbf{R}^n} \left| \int_{B_{|x|/2}} \frac{|y|}{|x|} |A(D)u(y)| dy \right|^q \frac{1}{|x|^n} dx \right)^{\frac{1}{q}}. \end{aligned}$$

By the Minkowski inequality (see for example [18, theorem 2.4]), we get

$$\begin{aligned} & \left( \int_{\mathbf{R}^n} \left| \int_{B_{|x|/2}} \frac{|y|}{|x|} |A(D)u(y)| dy \right|^q \frac{1}{|x|^n} dx \right)^{\frac{1}{q}} \\ &\leq \int_{\mathbf{R}^n} |y| |A(D)u(y)| \left( \int_{\mathbf{R}^n \setminus B_{2|y|}} \frac{1}{|x|^{n+q}} dx \right)^{\frac{1}{q}} dy \leq C' \int_{\mathbf{R}^n} |A(D)u|. \end{aligned}$$

For  $G_2$ , by the Minkowski inequality again,

$$\begin{aligned} & \left( \int_{\mathbf{R}^n} \left| \int_{\mathbf{R}^n} G_2(x, y) [A(D)u(y)] dy \right|^q \frac{1}{|x|^{n-(n-\ell)q}} dx \right)^{\frac{1}{q}} \\ &\leq \int_{\mathbf{R}^n} \left( \int_{\mathbf{R}^n} \frac{|G_2(x, y)|^q}{|x|^{n-(n-\ell)q}} dx \right)^{\frac{1}{q}} |A(D)u(y)| dy. \end{aligned}$$

If  $x \neq 0$ , since  $G_2(x, \cdot)$  is continuously differentiable on  $B_{|x|/2}$  and  $G$  is homogeneous of degree  $-(n-\ell)$ , if  $|x| \geq 2|y|$

$$|G_2(x, y)| \leq C \frac{|y|}{|x|^{n-\ell+1}};$$

while if  $|x| < 2|y|$ ,

$$|G_2(x, y)| \leq C \frac{1}{|x-y|^{n-\ell}}.$$

Therefore, since  $q < \frac{n}{n-\ell}$ ,

$$\begin{aligned} & \int_{\mathbf{R}^n} \frac{|G_2(x, y)|^q}{|x|^{n-(n-\ell)q}} dx \\ &\leq C \left( \int_{B_{2|y|}} \frac{1}{|x-y|^{(n-\ell)q} |x|^{n-(n-\ell)q}} dx + \int_{\mathbf{R}^n \setminus B_{2|y|}} \frac{|y|^q}{|x|^{n+q}} dx \right) \leq C'. \end{aligned}$$

We conclude that

$$\begin{aligned} & \left( \int_{\mathbf{R}^n} \left| \int_{\mathbf{R}^n} G_2(x, y) [A(D)u(y)] dy \right|^q \frac{1}{|x|^{n-(n-\ell)q}} dx \right)^{\frac{1}{q}} \\ &\leq C' \int_{\mathbf{R}^n} |A(D)u|. \end{aligned}$$

This completes the proof of the proposition.  $\square$

We end this section with the proof of theorem 3.



*Proof of theorem 3.* By proposition 2.1, there exists  $C > 0$  such that for every  $u \in C_c^\infty(\mathbf{R}^n; V)$ ,

$$\int_{\mathbf{R}^n} \frac{|D^{k-n+1}u(x)|}{|x|^{n-1}} dx \leq C \int_{\mathbf{R}^n} |A(D)u|.$$

On the other hand, we have the classical estimate (see for example [24, §2.3 (18)])

$$|D^{k-n}u(0)| \leq C' \int_{\mathbf{R}^n} \frac{|D^{k-n+1}u(x)|}{|x|^{n-1}} dx, \quad (2.2)$$

which follows from the integration over  $\theta \in \mathbf{S}^{n-1}$ , of the inequality

$$|D^{k-n}u(0)| = \left| \int_{\mathbf{R}^+} \frac{d}{dr} (D^{k-n}u(r\theta)) dr \right| \leq \int_{\mathbf{R}^+} |D^{k-n+1}u(r\theta)| dr.$$

Therefore, we have

$$|D^{k-n}u(0)| \leq C'' \int_{\mathbf{R}^n} |A(D)u|.$$

Since this estimate is invariant under translation, the conclusion follows.  $\square$

### 3. NECESSITY OF THE CANCELLATION CONDITION

In this section, we prove that if the Hardy inequality holds true for an elliptic operator  $A(D)$ , then  $A(D)$  is canceling.

**Proposition 3.1.** *Let  $A(D)$  be a linear differential operator of order  $k$  on  $\mathbf{R}^n$  from  $V$  to  $E$ ,  $\ell \in \{1, \dots, \min(k, n-1)\}$  and  $q \in [1, \frac{n}{n-\ell}]$ . If  $A(D)$  is elliptic and if there exists  $C \in \mathbf{R}$  such that for every  $u \in C_c^\infty(\mathbf{R}^n; V)$*

$$\left( \int_{\mathbf{R}^n} \frac{|D^{k-\ell}u(x)|^q}{|x|^{n-(n-\ell)q}} dx \right)^{\frac{1}{q}} \leq C \int_{\mathbf{R}^n} |A(D)u|,$$

*then  $A(D)$  is cancelling.*

*Proof.* The proof follows the lines of the counterpart of the proposition for the Sobolev inequality [30, proposition 5.5]. Let  $e \in \bigcap_{\xi \in \mathbf{R}^n \setminus \{0\}} A(\xi)[V]$ . Let  $\psi$  be in the Schwartz class  $\mathcal{S}(\mathbf{R}^n)$  of rapidly decaying smooth functions be such that  $\widehat{\psi} = 1$  on a neighborhood of 0 and define the family  $(\rho_\lambda)_{\lambda \geq 1}$  in  $\mathcal{S}(\mathbf{R}^n)$  for  $\lambda \geq 1$  and  $x \in \mathbf{R}^n$  by

$$\rho_\lambda(x) = \lambda^n \psi(\lambda x) - \frac{1}{\lambda^n} \psi\left(\frac{x}{\lambda}\right).$$

The family  $(\rho_\lambda)_{\lambda \geq 1}$  is bounded uniformly in  $L^1(\mathbf{R}^n)$  and for every  $\lambda \geq 1$ ,  $\widehat{\rho}_\lambda$  vanishes on a neighborhood of 0.

We then define a sequence  $(u_\lambda)_{\lambda \geq 1}$  in  $\mathcal{S}(\mathbf{R}^n, V)$  in such a way that for every  $\lambda \geq 1$  and  $\xi \in \mathbf{R}^n$ ,

$$\widehat{u}_\lambda(\xi) = (-i)^k \widehat{\rho}_\lambda(\xi) (A(\xi)^* \circ A(\xi))^{-1} [A(\xi)^*[e]].$$

Since  $A(D)$  is elliptic and homogeneous of order  $k$ ,  $u_\lambda$  is well defined as an element of  $\mathcal{S}(\mathbf{R}^n, V)$  and moreover, since  $e \in \bigcap_{\xi \in \mathbf{R}^n \setminus \{0\}} A(\xi)[V]$ ,  $A(D)u_\lambda = \rho_\lambda e$ . By a classical approximation argument and our assumption, we have for every  $\lambda \geq 1$ ,

$$\left( \int_{\mathbf{R}^n} \frac{|D^{k-\ell}u_\lambda(x)|^q}{|x|^{n-(n-\ell)q}} dx \right)^{\frac{1}{q}} \leq C \int_{\mathbf{R}^n} |A(D)u_\lambda|.$$

If  $G$  is the Green function given by lemma 2.2, this reads as

$$\left( \int_{\mathbf{R}^n} \frac{|(G * \rho_\lambda)(x)[e]|^q}{|x|^{n-(n-\ell)q}} dx \right)^{\frac{1}{q}} \leq C \int_{\mathbf{R}^n} |\rho_\lambda e| = C'. \quad (3.1)$$

We claim that for every  $x \in \mathbf{R}^n \setminus \{0\}$ ,  $\lim_{\lambda \rightarrow \infty} (G * \rho_\lambda)(x) = G(x)$ . Indeed, for every  $\lambda \geq 1$  and  $x \in \mathbf{R}^n \setminus \{0\}$ , we write

$$G * \rho_\lambda(x) - G(x) = \int_{\mathbf{R}^n} (G(x-y) - G(x)) \lambda^n \psi(\lambda y) dy - \int_{\mathbf{R}^n} G(x-y) \frac{1}{\lambda^n} \psi\left(\frac{y}{\lambda}\right) dy. \quad (3.2)$$

Since  $G$  is smooth on  $\mathbf{R}^n \setminus \{0\}$  and homogeneous of degree  $-(n-\ell)$ , there exists  $C > 0$  such that for every  $y \in B_{|x|/2}$ ,

$$|G(x-y) - G(x)| \leq \frac{C|y|}{|x|^{n-\ell+1}}.$$

Together with the fact that  $\psi$  belongs to  $\mathcal{S}(\mathbf{R}^n)$ , this implies that for every  $\alpha \in (0, n+1)$ ,

$$\begin{aligned} & \left| \int_{\mathbf{R}^n} (G(x-y) - G(x)) \lambda^n \psi(\lambda y) dy \right| \\ & \leq C'_\alpha \int_{B_{|x|/2}} \frac{|y|}{|x|^{n-\ell+1}} \frac{\lambda^n}{\lambda^\alpha |y|^\alpha} dy \\ & \quad + C'_\alpha \int_{\mathbf{R}^n \setminus B_{|x|/2}} \left( \frac{1}{|x-y|^{n-\ell}} + \frac{1}{|x|^{n-\ell}} \right) \frac{\lambda^n}{\lambda^\alpha |y|^\alpha} dy. \end{aligned}$$

This gives if  $\alpha > n$ ,

$$\left| \int_{\mathbf{R}^n} (G(x-y) - G(x)) \lambda^n \psi(\lambda y) dy \right| \leq \frac{C''}{\lambda^{\alpha-n} |x|^{\alpha-\ell}}.$$

It follows that the first term in the right hand side of (3.2) converges to 0. In order to estimate the second term, we pick  $\alpha \in (\ell, n)$  and write

$$\begin{aligned} \left| \int_{\mathbf{R}^n} G(x-y) \frac{1}{\lambda^n} \psi\left(\frac{y}{\lambda}\right) dy \right| & \leq C'_\alpha \int_{\mathbf{R}^n} \frac{1}{|x-y|^{n-\ell}} \frac{1}{\lambda^n} \frac{\lambda^\alpha}{|y|^\alpha} dy \\ & \leq C' \frac{1}{\lambda^{n-\alpha} |x|^{\alpha-\ell}}. \end{aligned}$$

This completes the proof of the fact that  $G * \rho_\lambda$  converges pointwisely to  $G$  on  $\mathbf{R}^n \setminus \{0\}$ .

By letting  $\lambda \rightarrow \infty$  in (3.1), we get by Fatou's Lemma

$$\left( \int_{\mathbf{R}^n} \frac{|G(x)[e]|^q}{|x|^{n-(n-\ell)q}} dx \right)^{\frac{1}{q}} < \infty.$$

Since  $G$  is homogeneous of degree  $-(n-\ell)$ , this implies that  $G(x)[e] = 0$  for every  $x \neq 0$ . In view of the properties of  $G$ , we thus have  $e \in \bigcap_{\xi \in \mathbf{R}^n \setminus \{0\}} \ker A(\xi)^*$ . Since  $e \in \bigcap_{\xi \in \mathbf{R}^n \setminus \{0\}} A(\xi)[V]$ , we conclude that  $e = 0$  and this completes the proof of proposition 3.1.  $\square$

## 4. NECESSITY OF THE ELLIPTICITY CONDITION

**4.1. The Hardy–Sobolev inequality.** The necessity of the ellipticity condition in theorem 4 for the inhomogeneous inequality corresponds to the case  $p = 1$  in the following proposition.

**Proposition 4.1.** *Let  $A(D)$  be a linear differential operator of order  $k$  on  $\mathbf{R}^n$  from  $V$  to  $E$ ,  $p \in [1, n)$  and  $q \in (p, \frac{np}{n-p}]$ . If there exists  $C \in \mathbf{R}$  such that for every  $u \in C_c^\infty(\mathbf{R}^n; V)$ ,*

$$\left( \int_{\mathbf{R}^n} \frac{|D^{k-1}u(x)|^q}{|x|^{n-(\frac{n}{p}-1)q}} dx \right)^{\frac{p}{q}} \leq C \int_{\mathbf{R}^n} |A(D)u|^p,$$

*then  $A(D)$  is elliptic.*

*Proof.* We assume by contradiction that there exist  $v \in V \setminus \{0\}$  and  $\xi \in \mathbf{R}^n \setminus \{0\}$  such that  $A(\xi)[v] = 0$ . Without loss of generality, we can further assume that  $|\xi| = 1$ . We fix  $\varphi \in C_c^\infty(\mathbf{R}^n)$  and  $\psi \in C_c^\infty(\mathbf{R})$  and we define for  $\lambda > 0$  the function  $u_\lambda \in C_c^\infty(\mathbf{R}^n; V)$  for every  $x \in \mathbf{R}^n$  by

$$u_\lambda(x) = \varphi\left(\frac{x}{\lambda}\right)\psi(\xi \cdot x)v.$$

By the iterated Leibniz product rule for differentiation, if  $\lambda \geq 1$  and  $x \in \mathbf{R}^n$ ,

$$\begin{aligned} \left| D^{k-1}u_\lambda(x) - \varphi\left(\frac{x}{\lambda}\right)D^{k-1}[\psi(\xi \cdot x)]v \right| &\leq \frac{C'}{\lambda} \left( \sum_{j=1}^k \left| D^j \varphi\left(\frac{x}{\lambda}\right) \right| \right) \left( \sum_{j=0}^{k-1} |D^j \psi(\xi \cdot x)| \right) \\ &\leq \frac{C'}{\lambda} \theta\left(\frac{x}{\lambda}\right) \eta(\xi \cdot x), \end{aligned}$$

where we have introduced to alleviate notation the functions  $\theta \in C_c(\mathbf{R}^n)$  and  $\eta \in C_c(\mathbf{R})$  defined for  $y \in \mathbf{R}^n$  by  $\theta(y) = \sum_{j=1}^k |D^j \varphi(y)|$  and for  $t \in \mathbf{R}$  by  $\eta(t) = \sum_{j=0}^{k-1} |D^j \psi(t)|$ . By the Minkowski inequality and our assumption, we thus get

$$\begin{aligned} &\left( \int_{\mathbf{R}^n} \frac{|\varphi(\frac{x}{\lambda})D^{k-1}\psi(\xi \cdot x)|^q}{|x|^{n-(\frac{n}{p}-1)q}} dx \right)^{\frac{1}{q}} \\ &\leq C \left( \int_{\mathbf{R}^n} |A(D)u_\lambda|^p \right)^{\frac{1}{p}} + C' \left( \int_{\mathbf{R}^n} \frac{|\theta(\frac{x}{\lambda})\eta(\xi \cdot x)|^q}{\lambda^q |x|^{n-(\frac{n}{p}-1)q}} dx \right)^{\frac{1}{q}}. \quad (4.1) \end{aligned}$$

Since  $A(\xi)[v] = 0$ , we have for every  $x \in \mathbf{R}^n$ ,

$$|A(D)u_\lambda(x)| \leq \frac{C''}{\lambda} \left( \sum_{j=1}^k \left| D^j \varphi\left(\frac{x}{\lambda}\right) \right| \right) \left( \sum_{j=0}^{k-1} |D^j \psi(\xi \cdot x)| \right) = \frac{C''}{\lambda} \theta\left(\frac{x}{\lambda}\right) \eta(\xi \cdot x).$$

If  $P_\xi$  denotes the orthogonal projection on  $\xi^\perp$  defined for  $y \in \mathbf{R}^n$  by  $P_\xi(y) = y - (\xi \cdot y)\xi$ , we obtain by a change of variable,

$$\begin{aligned} \int_{\mathbf{R}^n} |A(D)u_\lambda|^p &\leq \frac{C''^p}{\lambda^p} \int_{\mathbf{R}^n} \left| \theta\left(\frac{x}{\lambda}\right) \eta(\xi \cdot x) \right|^p dx \\ &= C''^p \lambda^{n-1-p} \int_{\mathbf{R}^n} \left| \theta\left(\frac{\xi \cdot y}{\lambda} \xi + P_\xi(y)\right) \eta(\xi \cdot y) \right|^p dy. \end{aligned}$$

If we choose  $R > 0$  in such a way that  $\text{supp } \theta \subset B_R$  and  $\text{supp } \eta \subset (-R, R)$ , then for every  $\lambda \geq 1$  and  $y \in \mathbf{R}^n$ ,

$$\left| \theta \left( \frac{\xi \cdot y}{\lambda} \xi + P_\xi(y) \right) \eta(\xi \cdot y) \right|^p \leq \|\theta\|_{L^\infty(\mathbf{R}^n)}^p \|\eta\|_{L^\infty(\mathbf{R})}^p \chi_{(-R,R)}(\xi \cdot y) \chi_{B_R}(P_\xi(y)),$$

so that, by comparison of integrals, for every  $\lambda \geq 1$ ,

$$\int_{\mathbf{R}^n} |A(D)u_\lambda|^p \leq C''' \lambda^{n-p-1}. \quad (4.2)$$

By the same changes of variables, we also get for every  $\lambda \geq 1$ ,

$$\int_{\mathbf{R}^n} \frac{|\theta(\frac{x}{\lambda})\eta(\xi \cdot x)|^q}{\lambda^q |x|^{n-(\frac{n}{p}-1)q}} dx = \lambda^{(\frac{n}{p}-1)q-1} \int_{\mathbf{R}^n} \frac{|\theta(\frac{\xi \cdot y}{\lambda} \xi + P_\xi(y))\eta(\xi \cdot y)|^q}{\lambda^q |\frac{\xi \cdot y}{\lambda} \xi + P_\xi(y)|^{n-(\frac{n}{p}-1)q}} dy.$$

The integrand can be bounded for every  $\alpha \in [0, n - (\frac{n}{p} - 1)q]$ ,  $\lambda > 0$  and  $y \in \mathbf{R}^n$  as

$$\begin{aligned} & \frac{|\theta(\frac{\xi \cdot y}{\lambda} \xi + P_\xi(y))\eta(\xi \cdot y)|^q}{\lambda^q |\frac{\xi \cdot y}{\lambda} \xi + P_\xi(y)|^{n-(\frac{n}{p}-1)q}} \\ & \leq \frac{\|\theta\|_{L^\infty(\mathbf{R}^n)} \|\eta\|_{L^\infty(\mathbf{R})} \chi_{(-R,R)}(\xi \cdot y) \chi_{B_R}(P_\xi(y))}{\lambda^{q-\alpha} |\xi \cdot y|^\alpha |P_\xi(y)|^{n(1-\frac{q}{p})+q-\alpha}}. \end{aligned}$$

If  $1 - (\frac{n}{p} - 1)q < \alpha < 1$ , the right-hand side is integrable, and thus

$$\int_{\mathbf{R}^n} \frac{|\theta(\frac{x}{\lambda})\eta(\xi \cdot x)|^q}{\lambda^q |x|^{n-(\frac{n}{p}-1)q}} dx \leq C'''' \lambda^{(\frac{n}{p}-1)q-1-(q-\alpha)}. \quad (4.3)$$

Finally by Fatou's lemma, we have

$$\begin{aligned} & \liminf_{\lambda \rightarrow \infty} \lambda^{1-(\frac{n}{p}-1)q} \int_{\mathbf{R}^n} \frac{|\varphi(\frac{x}{\lambda}) D^{k-1} \psi(\xi \cdot x)|^q}{|x|^{n-(\frac{n}{p}-1)q}} dx \\ & = \liminf_{\lambda \rightarrow \infty} \int_{\mathbf{R}^n} \frac{|\varphi(\frac{\xi \cdot y}{\lambda} \xi + P_\xi(y)) D^{k-1} \psi(\xi \cdot y)|^q}{|\frac{\xi \cdot y}{\lambda} \xi + P_\xi(y)|^{n-(\frac{n}{p}-1)q}} dy \\ & \geq \int_{\mathbf{R}^n} \frac{|\varphi(P_\xi(y)) D^{k-1} \psi(\xi \cdot y)|^q}{|P_\xi(y)|^{n-(\frac{n}{p}-1)q}} dy. \quad (4.4) \end{aligned}$$

By inserting (4.2) and (4.3) into (4.1) and letting  $\lambda \rightarrow +\infty$ , we deduce since  $q > p$  and  $\alpha < 1 < q$ , in view of (4.4)

$$\int_{\mathbf{R}^n} \frac{|\varphi(P_\xi(y)) D^{k-1} \psi(\xi \cdot y)|^q}{|P_\xi(y)|^{n-(\frac{n}{p}-1)q}} dy = 0.$$

This yields the desired contradiction because  $\psi$  and  $\varphi$  are arbitrary test functions.  $\square$

**4.2. The pure Hardy inequality.** In this section, we investigate the limiting case  $p = q$  in proposition 4.1.

4.2.1. *Condition on the nonellipticity set.* If  $A(D)$  is not elliptic, we can consider the set of those  $\xi \in \mathbf{R}^n$  such that  $A(\xi)$  is not one-to-one. We show that if a Hardy inequality holds for  $A(D)$ , then this set does not contain any linear subspace of dimension  $\lceil n - p \rceil$ .

**Proposition 4.2.** *Let  $A(D)$  be a linear differential operator of order  $k$  on  $\mathbf{R}^n$  from  $V$  to  $E$ . Let  $p \in [1, n]$ . If there exists  $C > 0$  such that for every  $u \in C_c^\infty(\mathbf{R}^n; V)$ ,*

$$\int_{\mathbf{R}^n} \frac{|D^{k-1}u(x)|^p}{|x|^p} dx \leq C \int_{\mathbf{R}^n} |A(D)u|^p,$$

*then for every linear subspace  $\Pi \subseteq \mathbf{R}^n$  such that  $\dim \Pi \geq n - p$ , there exists  $\xi \in \Pi$  such that  $A(\xi)$  is one-to-one.*

The above proposition implies in particular that the inequality cannot hold when  $\dim V > \dim E$ . It also shows that when  $n = 2$ , the operator  $A(D)$  is necessarily elliptic.

In order to prove proposition 4.2, we rely on the following algebra property.

**Lemma 4.3.** *Let  $A(D)$  be a homogenous differential operator from  $V$  to  $E$ . Then there exists a homogeneous differential operator  $B(D)$  from  $V$  to  $V$  such that*

$$A(D) \circ B(D) = 0$$

and

$$\max_{\xi \in \mathbf{R}^n} \dim B(\xi)[V] = \min_{\xi \in \mathbf{R}^n} \dim \ker A(\xi).$$

*Proof.* Choose  $\xi_* \in \mathbf{R}^n$  such that  $\dim \ker A(\xi_*) = s := \min_{\xi \in \mathbf{R}^n} \dim \ker A(\xi)$ . By the fundamental theorem of linear algebra,  $s = \max_{\xi \in \mathbf{R}^n} \dim A(\xi)^*[E]$ . Define  $P : V \rightarrow V$  to be a projection on  $A(\xi_*)^*[E]$  and choose the vectors  $e_1, \dots, e_s \in E$  so that their images  $A(\xi_*)^*[e_1], \dots, A(\xi_*)^*[e_s]$  are linearly independent in  $V$ . Define now  $B(\xi) \in \mathcal{L}(V; V)$  by

$$\begin{aligned} B(\xi)^*[v] &= \det(P[A(\xi)^*[e_1]], \dots, P[A(\xi)^*[e_s]])v \\ &\quad + \sum_{i=1}^s (-1)^i \det(P[v], P[A(\xi)^*[e_1]], \dots, P[A(\xi)^*[e_{i-1}]], \\ &\quad P[A(\xi)^*[e_{i+1}]], \dots, P[A(\xi)^*[e_s]]) A(\xi)^*[e_i] \end{aligned}$$

where  $\det$  is a determinant on  $A(\xi_*)^*[E] = P(V)$ .

For every  $e_0 \in E$  and  $v \in V$ , we have

$$\begin{aligned} e_0 \cdot A(\xi)[B(\xi)[v]] &= \sum_{i=0}^s (-1)^i \det(P[A(\xi)^*[e_0]], \dots, P[A(\xi)^*[e_{i-1}]], \\ &\quad P[A(\xi)^*[e_{i+1}]], \dots, P[A(\xi)^*[e_s]]) A(\xi)^*[e_i] \cdot v. \end{aligned}$$

Since the right hand side is antisymmetric with respect to the vectors  $A(\xi)^*[e_0], \dots, A(\xi)^*[e_s]$  and  $\dim A(\xi)^*[E] \leq s$ , we have  $e_0 \cdot A(\xi)[B(\xi)[v]] = 0$  and, since  $e_0 \in E$  is arbitrary,  $A(\xi)[B(\xi)[v]] = 0$ . In particular, we have for every  $\xi \in \mathbf{R}^n$ ,  $\dim B(\xi)[V] \leq \dim \ker A(\xi)$ . Since  $B(\xi_*)^*$  is one-to-one on  $\ker P$  and  $\dim \ker P = \dim \ker A(\xi_*)$ , we have  $B(\xi_*)^*[V] = \ker P$  and  $\dim B(\xi_*)[V] = \dim \ker A(\xi_*)$ .

Finally, we claim that  $P(B(\xi)^*[v]) = 0$  for every  $v \in V$  and every  $\xi \in \mathbf{R}^n$ . Indeed, let  $w_0 = P[v]$  and  $w_i = P(A(\xi)^*[e_i])$  for  $i \in \{1, \dots, s\}$ . Then

$$P(B(\xi)^*[v]) = \sum_{i=0}^s (-1)^i \det(w_0, \dots, w_{i-1}, w_{i+1}, \dots, w_s) w_i.$$

Since  $w_i \in P[V]$  for  $i = 0, \dots, s$  and  $\dim P[V] = s$ , we get  $P(B(\xi)^*[v]) = 0$  which proves the claim. This implies that  $B(\xi)^*[V] \subseteq \ker P = B(\xi_*)^*[V]$ . Hence,

$$\max_{\xi \in \mathbf{R}^n} \dim B(\xi)^*[V] = \dim B(\xi_*)^*[V].$$

Since  $\dim B(\xi)^*[V] = \dim B(\xi)[V]$ , this completes the proof of the lemma.  $\square$

*Proof of proposition 4.2.* Consider a linear subspace  $\Pi \subseteq \mathbf{R}^n$  such that for every  $\xi \in \Pi$ ,  $\text{rank } A(\xi) < \dim V$ . Without loss of generality, one may assume that  $\Pi = \mathbf{R}^m \times \{0\}$ . We introduce the linear differential operator on  $\mathbf{R}^m$  from  $V$  to  $E$  defined for  $\xi' \in \mathbf{R}^m$  by  $A'(\xi') := A(\xi', 0)$ . By lemma 4.3, there exists a linear differential operator  $B'(D')$  on  $\mathbf{R}^m$  from  $V$  to  $V$  such that  $B'(D') \neq 0$  and  $A'(D')B'(D') = 0$ .

Let  $v \in C_c^\infty(\mathbf{R}^m; V)$  be such that  $w := B'(D')v \neq 0$ . For any  $\varphi \in C_c^\infty(\mathbf{R}^{n-m})$  and  $\lambda > 0$ , we consider  $u_\lambda \in C^\infty(\mathbf{R}^n; V)$  defined for  $(x', x'') \in \mathbf{R}^m \times \mathbf{R}^{n-m}$  by  $u_\lambda(x', x'') = \varphi(x'')w(\lambda x')$ . Since  $A'(D')B'(D')v = 0$ , we get

$$|A(D)u_\lambda(x', x'')| \leq C' \sum_{j=0}^{k-1} \lambda^j |D^j w(\lambda x')| |D^{k-j} \varphi(x'')|.$$

Whence,

$$\limsup_{\lambda \rightarrow \infty} \frac{1}{\lambda^{p(k-1)-m}} \int_{\mathbf{R}^n} |A(D)u_\lambda|^p \leq C' \int_{\mathbf{R}^n} |D^{k-1} w(x')|^p |D \varphi(x'')|^p dx' dx''.$$

By definition of  $u_\lambda$ , we have

$$\lambda^{p(k-1)-m} \int_{\mathbf{R}^n} \frac{|D^{k-1} w(x')|^p |\varphi(x'')|^p}{\frac{|x'|^p}{\lambda^p} + |x''|^p} dx' dx'' \leq C'' \int_{\mathbf{R}^n} \frac{|D^{k-1} u_\lambda(x)|^p}{|x|^p} dx,$$

By the assumption applied to  $u_\lambda$ , we thus get

$$\begin{aligned} \limsup_{\lambda \rightarrow \infty} \int_{\mathbf{R}^n} \frac{|D^{k-1} w(x')|^p |\varphi(x'')|^p}{\frac{|x'|^p}{\lambda^p} + |x''|^p} dx' dx'' \\ \leq C''' \int_{\mathbf{R}^n} |D^{k-1} w(x')|^p |D \varphi(x'')|^p dx' dx''. \end{aligned}$$

We let  $\lambda$  go to  $+\infty$  and then use Fubini theorem to obtain

$$\int_{\mathbf{R}^{n-m}} \frac{|\varphi(x'')|^p}{|x''|^p} dx'' \leq C'''' \int_{\mathbf{R}^{n-m}} |D \varphi(x'')|^p dx''.$$

Since this must be true for any  $\varphi \in C_c^\infty(\mathbf{R}^{n-m})$ , this implies that  $p < n - m$ .  $\square$

**4.2.2. Dimension reduction.** In proposition 4.2, we dealt with failure of the ellipticity because  $A(\xi)$  is not one-to-one. The ellipticity can fail more boldly when  $A(\xi) = 0$  on a  $(n - m)$ -dimensional plane. In this case, the validity of the inequality reduces to that of an inequality on the  $m$ -dimensional space.

**Proposition 4.4.** *Let  $A(D)$  be a linear differential operator of order  $k$  on  $\mathbf{R}^n$  from  $V$  to  $E$ . We assume that there exists a vector subspace  $\Pi \subseteq \mathbf{R}^n$  of dimension  $m \in \{1, \dots, n-1\}$  and a linear differential operator  $A'(D')$  of order  $k$  on  $\Pi$  from  $V$  to  $E$  such that for any  $\xi \in \mathbf{R}^n$ , we have  $A(\xi) = A'(P(\xi))$ , where  $P : \mathbf{R}^n \rightarrow \Pi$  is a linear projection onto  $\Pi$ . Let  $\ell \in \{0, \dots, k\}$  and  $C \in \mathbf{R}$ .*

*For every  $u \in C_c^\infty(\mathbf{R}^n; V)$ ,*

$$\int_{\mathbf{R}^n} \frac{|D^{k-\ell}u(x)|^p}{|x|^{\ell p}} dx \leq C \int_{\mathbf{R}^n} |A(D)u|^p,$$

*if and only if  $k = \ell$  and for every  $u \in C_c^\infty(\Pi; V)$ ,*

$$\int_{\Pi} \frac{|u(x)|^p}{|x|^{kp}} dx \leq C \int_{\Pi} |A'(D')u|^p.$$

This proposition generalizes example (1.5) given in the introduction.

*Proof of proposition 4.4.* Without loss of generality, we can assume that  $\Pi = \mathbf{R}^m \times \{0\}$  and  $P(x) = x'$  where we write  $x = (x', x'') \in \mathbf{R}^m \times \mathbf{R}^{n-m}$ .

Assume first that the inequality on  $\mathbf{R}^n$  holds true. For every function  $v \in C_c^\infty(\mathbf{R}^m; V)$ ,  $w \in C_c^\infty(\mathbf{R}^{n-m}; V) \setminus \{0\}$  and  $\lambda > 0$ , we consider the map  $u_\lambda(x', x'') = v(x')w(\lambda x'')$ . We observe that  $A(D)u_\lambda(x', x'') = (A'(D')v)(x')w(\lambda x'')$  and  $|D^{k-\ell}u_\lambda(x', x'')| \geq \lambda^{k-\ell}|v(x')||D^{k-\ell}w(\lambda x'')|$ . By inserting this in the inequality on  $\mathbf{R}^n$ , we get

$$\begin{aligned} \lambda^{k-\ell} \int_{\mathbf{R}^n} \frac{|v(x')|^p |D^{k-\ell}w(x'')|^p}{|(x', x''/\lambda)|^{\ell p}} dx' dx'' \\ \leq C \int_{\mathbf{R}^n} |A'(D')v(x')|^p |w(x'')|^p dx' dx''. \end{aligned}$$

Then, necessarily,  $k = \ell$  and by letting  $\lambda$  to  $+\infty$ , we get

$$\int_{\mathbf{R}^n} \frac{|v(x')w(x'')|^p}{|x'|^{kp}} dx' dx'' \leq C \int_{\mathbf{R}^n} |A'(D')v(x')|^p |w(x'')|^p dx' dx'',$$

and the inequality on  $\Pi$  now follows from Fubini theorem.

Conversely, assume that  $k = \ell$  and that the inequality holds on  $\Pi$ . For every  $u \in C_c^\infty(\mathbf{R}^n; V)$ , we have by assumption

$$\int_{\mathbf{R}^m} \frac{|u(x', x'')|^p}{|x'|^{kp}} dx' \leq C \int_{\mathbf{R}^m} |A'(D')u(x', x'')|^p dx'.$$

It follows that

$$\int_{\mathbf{R}^m} \frac{|u(x', x'')|^p}{|(x', x'')|^{kp}} dx' \leq C \int_{\mathbf{R}^m} |A(D)u(x', x'')|^p dx'.$$

We now integrate in  $x''$  to get the result.  $\square$

## 5. HARDY INEQUALITIES FOR NONELLIPTIC FAMILIES OF OPERATORS

**5.1. Direct sum of directional derivatives.** We proceed to give a necessary and sufficient condition for a special class of differential operators of order one:

**Proposition 5.1.** *Let  $\ell \in \mathbf{N}_*$ ,  $a_1, \dots, a_\ell \in V \setminus \{0\}$  and  $b_1, \dots, b_\ell \in \mathbf{R}^n \setminus \{0\}$ . The following are equivalent*

(i) *there exists  $C > 0$  such that for every  $u \in C_c^\infty(\mathbf{R}^n; V)$ ,*

$$\int_{\mathbf{R}^n} \frac{|u(x)|}{|x|} dx \leq C \sum_{i=1}^{\ell} \int_{\mathbf{R}^n} |a_i \cdot Du(x)[b_i]| ,$$

(ii) *for every  $\xi \in \mathbf{R}^n \setminus \{0\}$  and  $v \in V \setminus \{0\}$  there exists  $i \in \{1, \dots, \ell\}$  such that  $(a_i \cdot v) \neq 0$  and  $|b_i|^2 \xi \neq (\xi \cdot b_i) b_i$ .*

If the linear differential operator  $A(D)$  of order 1 on  $\mathbf{R}^n$  from  $V$  to  $\mathbf{R}^\ell$  is defined for  $v \in V$  and  $\xi \in \mathbf{R}^n$  by

$$A(\xi)(v) := ((\xi \cdot b_i)(a_i \cdot v))_{1 \leq i \leq \ell},$$

one checks that  $A(D)$  is canceling if and only if  $n \geq 2$  and that  $A(D)$  is elliptic if and only if for every  $\zeta \in \mathbf{R}^n \setminus \{0\}$  and  $v \in \mathbf{R}^n \setminus \{0\}$  there exists  $i \in \{1, \dots, \ell\}$  such that  $(a_i \cdot v) \neq 0$  and  $\zeta \cdot b_i \neq 0$ , that is, instead of forbidding the vector  $b_i$  to be colinear with  $\xi$ , we are prohibiting  $b_i$  from being orthogonal to  $\zeta$ . In the two dimensional case, the ellipticity condition is seen to be equivalent to (ii) by taking  $(\zeta \cdot \xi) = 0$ , in higher dimension, the condition (ii) is weaker than the ellipticity.

For instance, when  $\ell = \dim V + 1$ , (ii) is satisfied if and only if  $a_1, \dots, a_\ell$  are  $\ell - 1$  by  $\ell - 1$  linearly independent and  $b_1, \dots, b_\ell$  are 2 by 2 linearly independent.

*Proof.* Assume by contradiction that (i) holds while (ii) is not satisfied. Then there exist  $\xi \in \mathbf{R}^n \setminus \{0\}$  and  $v \in V \setminus \{0\}$  such that for every  $i \in \{1, \dots, \ell\}$ , either  $v \cdot a_i = 0$  or  $|b_i|^2 \xi = (b_i \cdot \xi) b_i$ .

For every  $\varphi \in C_c^\infty(\mathbf{R})$  and  $\psi \in C_c^\infty(\mathbf{R}^n)$ , define

$$u_\lambda(x) = \varphi(\xi \cdot x) \psi(\lambda(|\xi|^2 x - (\xi \cdot x) \xi)) v$$

and note that

$$\begin{aligned} a_i \cdot Du_\lambda(x)[b_i] &= (v \cdot a_i)(\xi \cdot b_i) \varphi'(\xi \cdot x) \psi(\lambda(|\xi|^2 x - (\xi \cdot x) \xi)) \\ &\quad + (v \cdot a_i) \varphi(\xi \cdot x) \lambda D\psi(\lambda(|\xi|^2 x - (\xi \cdot x) \xi)) [|\xi|^2 b_i - (\xi \cdot b_i) \xi]. \end{aligned}$$

Now apply (i) to  $u_\lambda$

$$\begin{aligned} \int_{\mathbf{R}^n} \frac{|\varphi(\xi \cdot x) \psi(\lambda(|\xi|^2 x - (\xi \cdot x) \xi))|}{|x|} dx \\ \leq C \int_{\mathbf{R}^n} |\varphi'(\xi \cdot x) \psi(\lambda(|\xi|^2 x - (\xi \cdot x) \xi))| dx , \end{aligned}$$



By a change of variable, this becomes

$$\begin{aligned} \int_{\mathbf{R}^n} \frac{|\varphi(\xi \cdot x) \psi(|\xi|^2 x - (\xi \cdot x) \xi)|}{\sqrt{|\xi \cdot x|^2 + \lambda^{-2}(|x|^2 |\xi|^2 - |\xi \cdot x|^2)}} dx \\ \leq C \int_{\mathbf{R}^n} |\varphi'(\xi \cdot x) \psi(|\xi|^2 x - (\xi \cdot x) \xi)| dx, \end{aligned}$$

By letting  $\lambda \rightarrow \infty$ , we conclude that for every  $\varphi \in C_c^\infty(\mathbf{R})$ ,

$$\int_{\mathbf{R}} \frac{|\varphi(t)|}{|t|} dt \leq C \int_{\mathbf{R}} |\varphi'(t)| dt,$$

which cannot hold.

Conversely, if (ii) holds true, then for every  $\xi \in \mathbf{R}^n \setminus \{0\}$

$$\{a_i : i \in \{1, \dots, \ell\} \text{ and } |b_i|^2 \xi \neq (\xi \cdot b_i) b_i\} \quad (5.1)$$

generates  $V$ . In particular, the set

$$\{a_i : i \in \{1, \dots, \ell\}\}$$

generates  $V$ . Without loss of generality, we can assume that  $|b_i| = 1$  for every  $i \in \{1, \dots, \ell\}$ .

In order to prove the corresponding Hardy inequality (i), it is thus enough to establish that for any  $u \in C_c^\infty(\mathbf{R}^n; V)$  and  $1 \leq i \leq \ell$

$$\int_{\mathbf{R}^n} \frac{|a_i \cdot u(x)|}{|x|} dx \leq C \sum_{j=1}^{\ell} \int_{\mathbf{R}^n} |a_j \cdot Du(x)[b_j]| dx.$$

Consider the case  $i = 1$  and let  $u \in C_c^\infty(\mathbf{R}^n; V)$ . By taking  $\xi = b_1$  in (5.1), we can write  $a_1 = \lambda_2 a_{i_2} + \dots + \lambda_r a_{i_r}$ , for some  $r \in \{2, \dots, \ell\}$ ,  $i_j \in \{2, \dots, \ell\}$ ,  $\lambda_j \in \mathbf{R}$  and  $|b_1 \cdot b_{i_j}| < 1$  for every  $j \in \{2, \dots, r\}$ . In order to simplify the notation, we can assume that  $(i_2, \dots, i_r) = (2, \dots, r)$ . It follows that

$$a_1 \cdot u(x) = \sum_{j=2}^r \lambda_j a_j \cdot u(x). \quad (5.2)$$

We now estimate

$$\int_{\mathbf{R}^n} \frac{|a_1 \cdot u(x)|}{|x|} dx \leq \int_{\mathbf{R}^n \setminus \bigcup_{i=2}^r B_i} \frac{|a_1 \cdot u(x)|}{|x|} dx + \sum_{i=2}^r \int_{B_i} \frac{|a_1 \cdot u(x)|}{|x|} dx,$$

where for  $i \in \{2, \dots, r\}$ ,

$$B_i = \{x \in \mathbf{R}^n : |x \cdot b_1| \leq |x \cdot b_i|\}.$$

By (5.2), this gives

$$\begin{aligned} \int_{\mathbf{R}^n} \frac{|a_1 \cdot u(x)|}{|x|} dx &\leq \int_{\mathbf{R}^n \setminus \bigcup_{i=2}^r B_i} \sum_{j=2}^r |\lambda_j| \frac{|a_j \cdot u(x)|}{|x|} dx + \sum_{i=2}^r \int_{B_i} \frac{|a_1 \cdot u(x)|}{|x|} dx \\ &\leq C \sum_{i=2}^r \int_{\mathbf{R}^n \setminus B_i} \frac{|a_i \cdot u(x)|}{|x|} dx + \sum_{i=2}^r \int_{B_i} \frac{|a_1 \cdot u(x)|}{|x|} dx, \end{aligned} \quad (5.3)$$

Since for every  $1 \leq i \leq r$ , the roles of  $i$  and 1 are symmetric, we only need to prove

$$\int_{B_i} \frac{|a_1 \cdot u(x)|}{|x|} dx \leq C \int_{\mathbf{R}^n} |a_1 \cdot Du(x)[b_1]| dx, \quad (5.4)$$

In view of the identity

$$a_1 \cdot u(x) = \int_{-\infty}^0 D(a_1 \cdot u)(x + tb_1)[b_1] dt, \quad (5.5)$$

we have

$$\int_{B_i} \frac{|a_1 \cdot u(x)|}{|x|} dx \leq \int_{B_i} \int_{-\infty}^0 |D(a_1 \cdot u)(x + tb_1)[b_1]| dt \frac{dx}{|x|}.$$

We complete the proof of the proposition by the next lemma.  $\square$

**Lemma 5.2.** *Let  $b, c \in \mathbf{R}^n \setminus \{0\}$  and define*

$$J = \{x \in \mathbf{R}^n : |b \cdot x| \leq |c \cdot x|\}.$$

*If  $|b \cdot c| < |b|^2$ , then for every nonnegative function  $f \in L^1(\mathbf{R}^n)$ ,*

$$\int_J \int_{\mathbf{R}} \frac{f(x + tb)}{|x|} dt dx \leq 2 \frac{|b| \sqrt{|b|^2 |c|^2 - (b \cdot c)^2}}{|b|^4 - (b \cdot c)^2} \int_{\mathbf{R}^n} f(x) dx.$$

*Proof.* By the change of variable formula, one has

$$\int_J dx \int_{\mathbf{R}} \frac{f(x + tb)}{|x|} dt = \int_{\mathbf{R}^n} dy \int_{D_y} \frac{f(y)}{|y - tb|} dt, \quad (5.6)$$

where

$$D_y = \{t \in \mathbf{R} : |b \cdot (y - tb)| \leq |c \cdot (y - tb)|\}.$$

One notes that for every  $y \in \mathbf{R}^n$  and  $t \in \mathbf{R}$ ,

$$|y - tb| \geq \left| y - \frac{b \cdot y}{|b|^2} b \right|$$

and that

$$D_y = \left\{ t \in \mathbf{R} : \left( \frac{(b - c) \cdot y}{|b|^2 - b \cdot c} - t \right) \left( t - \frac{(b + c) \cdot y}{|b|^2 + b \cdot c} \right) \geq 0 \right\},$$

so that by the Cauchy-Schwarz inequality,

$$\begin{aligned} |D_y| &= 2 \left| \frac{c \cdot (|b|^2 y - (b \cdot y)b)}{|b|^4 - (b \cdot c)^2} \right| \\ &= 2 \left| \frac{(|b|^2 c - (b \cdot c)b) \cdot (|b|^2 y - (b \cdot y)b)}{|b|^2 (|b|^4 - (b \cdot c)^2)} \right| \\ &\leq 2 \frac{\sqrt{|b|^2 |c|^2 - (b \cdot c)^2} |b|^2 |y - (b \cdot y)b|}{|b| (|b|^4 - (b \cdot c)^2)}. \end{aligned}$$

In view of (5.6), this implies

$$\begin{aligned} \int_J \int_{\mathbf{R}} \frac{f(x + tb)}{|x|} dt &\leq \int_{\mathbf{R}^n} \frac{|D_y|}{\left| y - \frac{b \cdot y}{|b|^2} b \right|} f(y) dy \\ &\leq 2 \frac{|b| \sqrt{|b|^2 |c|^2 - (b \cdot c)^2}}{|b|^4 - (b \cdot c)^2} \int_{\mathbf{R}^n} f(y) dy. \end{aligned}$$

This completes the proof of the lemma.  $\square$

**5.2. Direct sum of general differential operators.** We now generalize the sufficiency part of proposition 5.1 to a more general class of non elliptic operators.

**Proposition 5.3.** *Consider a linear differential operator  $A(D)$  of order 1 from  $V$  to  $E = E_1 \oplus \dots \oplus E_\ell$  which can be written for  $\xi \in \mathbf{R}^n$  as*

$$A(\xi) = \sum_{i=1}^{\ell} A_i(P_i(\xi)) \circ Q_i,$$

where  $P_i \in \mathcal{L}(\mathbf{R}^n; \mathbf{R}^n)$  and  $Q_i \in \mathcal{L}(V; V)$  are projections and  $A_i$  is an elliptic linear differential operator of order 1 on  $\Pi_i := P_i(\mathbf{R}^n)$  from  $V_i = Q_i(V)$  to  $E_i$ .

If  $\bigcap_{i=1}^{\ell} \ker Q_i = \{0\}$  and for every  $i \in \{1, \dots, \ell\}$ ,

$$\bigcap_{j \in I_i} \ker Q_j \subseteq \ker Q_i, \quad (5.7)$$

with  $I_i := \{j : \ker(P_i) \not\subseteq \ker(P_j) \text{ and } \ker(P_j) \not\subseteq \ker(P_i)\}$ ,

then there exists  $C > 0$  such that for every  $u \in C_c^\infty(\mathbf{R}^n; V)$ ,

$$\int_{\mathbf{R}^n} \frac{|u(x)|}{|x|} dx \leq C \sum_{i=1}^{\ell} \int_{\mathbf{R}^n} |A(D)u(x)| dx.$$

The assumption (5.7) implies that  $A(D)$  is canceling. Indeed, for every  $i \in \{1, \dots, \ell\}$ , either  $V_i = \{0\}$  or  $I_i \neq \emptyset$ . In the latter case,  $\ker(P_i) \neq \{0\}$ . Hence, there exists  $\xi \neq 0$  such that  $P_i(\xi) = 0$ , which implies that  $A(\xi)[V] \cap E_i = \{0\}$ . If  $V_i = \{0\}$ , this is true for any  $\xi \in \mathbf{R}^n$ . Since this holds for every  $i \in \{1, \dots, \ell\}$ ,  $A(D)$  is canceling.

This proposition coincides with proposition 5.1 in the particular case when  $P_i(\xi) = (\xi \cdot b_i)b_i$  and  $Q_i(v) = (a_i \cdot v)a_i$  for  $a_1, \dots, a_\ell \in V \setminus \{0\}$  and  $b_1, \dots, b_\ell \in \mathbf{R}^n \setminus \{0\}$ . We thus have  $V_i = \mathbf{R}a_i$  and  $\Pi_i = \mathbf{R}b_i$ . Indeed, the assumptions  $\bigcap_{i=1}^{\ell} \ker Q_i = \{0\}$  and  $\bigcap_{j \in I_i} \ker Q_j \subseteq \ker Q_i$  hold true if and only if the families  $\{a_i\}_{1 \leq i \leq \ell}$ ,  $\{b_i\}_{1 \leq i \leq \ell}$  satisfy assumption (ii) in proposition 5.1.

As an example, consider the linear differential operator  $A(D)$  on  $\mathbf{R}^4$  from  $\mathbf{R}^2$  to  $\mathbf{R}^4$  defined for  $\xi = (\xi_1, \dots, \xi_4)$  by

$$A(\xi) = \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \\ \xi_3 & -\xi_4 \\ \xi_4 & \xi_3 \end{pmatrix}.$$

This operator, which is not elliptic and cannot be considered in the framework of proposition 5.1, has the form described in proposition 5.3 with  $V_1 = \mathbf{R}(1, 0)$ ,  $V_2 = \mathbf{R}(0, 1)$ ,  $V_3 = \mathbf{R}^2$  and

$$A_1(\xi_1) \circ Q_1 = (\xi_1 \ 0), \quad A_2(\xi_2) \circ Q_2 = (0 \ \xi_2),$$

$$A_3(\xi_3, \xi_4) \circ Q_3 = \begin{pmatrix} \xi_3 & -\xi_4 \\ \xi_4 & \xi_3 \end{pmatrix}.$$

Assumption (5.7) is satisfied so that the Hardy inequality holds true in that case.

*Proof of proposition 5.3.* The proof is very similar to the proof of proposition 5.1. We only outline the main differences. Without loss of generality, we assume that  $V_i \neq \{0\}$  and  $\Pi_i \neq \{0\}$ . If  $\tilde{P}_i$  is the orthogonal projection on  $\ker(P_i)^\perp$ , then there exists a differential operator  $\tilde{A}_i$  of order 1 on  $\ker(P_i)^\perp$  from  $V_i$  to  $E_i$  such that  $\tilde{A}_i \circ \tilde{P}_i = A_i \circ P_i$ . By construction  $\ker \tilde{P}_i = \ker P_i$  and  $\tilde{A}_i$  is elliptic. We can thus assume without loss of generality that  $P_i$  is an orthogonal projection.

Since  $\bigcap_{i=1}^\ell \ker Q_i = \{0\}$ , there exists  $C > 0$  such that for every  $v \in V$ , we have

$$|v| \leq C \sum_{i=1}^\ell |Q_i(v)|.$$

Thus, we only need to prove that for every  $i \in \{1, \dots, \ell\}$ ,

$$\int_{\mathbf{R}^n} \frac{|Q_i(u)(x)|}{|x|} dx \leq C \sum_{j=1}^\ell \int_{\mathbf{R}^n} |A_j(P_j(D))Q_j(u)(x)| dx, \quad (5.8)$$

Consider the case  $i = 1$ . We define for  $j \in I_1$  the set  $B_j := \{x \in \mathbf{R}^n : |P_1(x)| \leq |P_j(x)|\}$ . Since  $\bigcap_{j \in I_1} \ker Q_j \subseteq \ker Q_1$ , there exists  $C > 0$  such that for every  $v \in V$ ,

$$|Q_1(v)| \leq C \sum_{j \in I_1} |Q_j(v)|.$$

By using this estimate on  $\mathbf{R}^n \setminus \bigcup_{j \in I_1} B_j$  exactly as in proposition 5.1 (see (5.3)) we are thus reduced to prove the analogue of (5.4), namely for every  $j \in I_1$

$$\int_{B_j} \frac{|Q_1(u)(x)|}{|x|} dx \leq C \int_{\mathbf{R}^n} |A_1(P_1(D))Q_1(u)(x)| dx. \quad (5.9)$$

Let  $n_1 = \dim \Pi_1$ . Consider first the case  $n_1 = 1$  : there exists  $b_1 \in \mathbf{R}^n$ ,  $|b_1| = 1$  and a linear map  $a_1 \in \mathcal{L}(V_1; E_1)$  such that for every  $v \in V_1$ ,

$$A_1(P(\xi))[v] = \xi \cdot b_1 a_1[v].$$

Since  $a_1$  is one-to-one, there exists  $C > 0$  (not depending on  $u$ ) such that for every  $x \in \mathbf{R}^n$

$$|Q_1(u)(x)| \leq C |a_1[Q_1(u)(x)]|$$

By the identity (5.5) applied to  $a_1(Q_1(u))$ , we thus get

$$\int_{B_j} \frac{|Q_1(u)(x)|}{|x|} dx \leq C \int_{B_j} \frac{dx}{|x|} \int_{-\infty}^0 |A_1(P_1(D))(Q_1(u))(x + tb_1)| dt.$$

By the change of variable formula, we get

$$\int_{B_j} \frac{|Q_1(u)(x)|}{|x|} dx \leq C \int_{\mathbf{R}^n} |A_1(P_1(D))(Q_1(u))(y)| dy \int_{J_j^y} \frac{dt}{|y - tb_1|} \quad (5.10)$$

where

$$J_j^y = \{t : |P_1(y - tb_1)| \leq |P_j(y - tb_1)|\}.$$

When  $n_1 \geq 2$ , we introduce the Green function  $G_1$  corresponding to  $A_1(P_1(D))$  on  $\Pi_1$  given by lemma 2.2, which is homogeneous of degree  $1 - n_1$ .

We write every  $x \in \mathbf{R}^n$  as  $x = (y, z) \in \Pi_1 \times \ker P_1$ :

$$\begin{aligned} & \int_{B_j} \frac{|Q_1(u)(x)|}{|x|} dx \\ & \leq C \int_{B_j} \frac{dy dz}{|(y, z)|} \int_{\Pi_1} \frac{1}{|y - t|^{n_1-1}} |A_1(P_1(D))Q_1(u)(t, z)| dt \\ & \leq C \int_{\mathbf{R}^n} |A_1(P_1(D))Q_1(u)(t, z)| dt dz \int_{B_j^z} \frac{dy}{|(y, z)||y - t|^{n_1-1}}, \end{aligned} \quad (5.11)$$

where

$$B_j^z = \{y \in \Pi_1 : (y, z) \in B_j\}.$$

In view of (5.10) and (5.11), proposition 5.3 then follows from the next lemma.  $\square$

**Lemma 5.4.** *There exists  $C > 0$  such that for every  $t \in \Pi_1$ , for every  $z \in \ker P_1$ ,*

$$\int_{B_j^z} \frac{dy}{|(y, z)||y - t|^{n_1-1}} \leq C. \quad (5.12)$$

*Proof.* We write  $\Pi_1 = (\Pi_1 \cap \Pi_j) \oplus \Pi'_1$ , where  $\Pi'_1 = \ker P_j \cap \Pi_1$ , and any  $y \in \Pi_1$  as  $y = y' + y'' \in \Pi'_1 \oplus (\Pi_1 \cap \Pi_j)$ . We thus have

$$|P_1(y, z)|^2 = |y|^2 = |y'|^2 + |y''|^2.$$

Since  $y'' \in \Pi_1 \cap \Pi_j$ ,  $z \in \ker P_1$  and  $P_j$  is an orthogonal projection, we have  $y'' \cdot P_j(z) = 0$ . This gives

$$|P_j(y, z)|^2 = |y'' + P_j(z)|^2 = |y''|^2 + |P_j(z)|^2.$$

Hence, the set  $B_j^z$  is a cylinder:

$$B_j^z = \{y \in \Pi_1 : |y'| \leq |P_j(z)|\}.$$

By a pointwise bound on the integrand, we have

$$\int_{B_j^z} \frac{dy}{|(y, z)||y - t|^{n_1-1}} \leq \int_{B_j^z} \frac{dy}{|z|^{1/2}|y|^{1/2}|y - t|^{n_1-1}}.$$

By a double application of the Hardy–Littlewood rearrangement inequality (see for example [18, theorem 3.4]),

$$\begin{aligned} \int_{B_j^z} \frac{dy}{|z|^{1/2}|y|^{1/2}|y - t|^{n_1-1}} & \leq \int_{B_j^z} \frac{dy}{|z|^{1/2}|y|^{1/2}|(y', y'' - t'')|^{n_1-1}} \\ & \leq \int_{B_j^z} \frac{dy}{|z|^{1/2}|y|^{n_1-1/2}}. \end{aligned}$$

Since  $\Pi_1 \not\subseteq \Pi_j$ , we have  $\dim \Pi_1 \cap \Pi_j < n_1$ , so that the right-hand side integral is finite. By homogeneity, we thus have

$$\int_{B_j^z} \frac{dy}{|z|^{1/2}|y|^{n_1-1/2}} = C' \frac{|P_j(z)|^{1/2}}{|z|^{1/2}}.$$

and the conclusion follows.  $\square$

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LABORATOIRE D'ANALYSE, TOPOLOGIE, PROBABILITÉS UMR7353, AIX-MARSEILLE  
UNIVERSITÉ, CMI 39, RUE FRÉDÉRIC JOLIOT CURIE, 13453 MARSEILLE CEDEX 13,  
FRANCE

*E-mail address:* bousquet@cmi.univ-mrs.fr

UNIVERSITÉ CATHOLIQUE DE LOUVAIN, INSTITUT DE RECHERCHE EN MATHÉMATIQUE  
ET PHYSIQUE (IRMP), CHEMIN DU CYCLOTRON 2 BTE L7.01.01, 1348 LOUVAIN-LA-  
NEUVE, BELGIUM

*E-mail address:* Jean.VanSchaftingen@uclouvain.be